4-3 Multiplying Matrices

Main Ideas

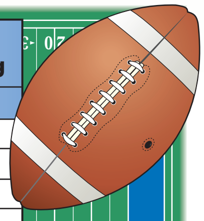
- Multiply matrices.
- Use the properties of matrix multiplication

GET READY for the Lesson

The table shows the scoring summary of the Carolina Panthers for the 2005 season. The team's record can be summarized in the record matrix R. The values for each type of score can be organized in the point values matrix P.

Carolina Panthers Regular Season Scoring

Type	Number
Touchdown	45
Extra Point	43
Field Goal	26
2-Point Conversion	1
Safety	0



Source: National Football League

Record

$$R = \begin{array}{l} \begin{bmatrix} 45 \\ 43 \\ 26 \\ 1 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \text{touchdown} \\ \text{extra point} \\ \text{field goal} \\ \text{2-point conversion} \\ \text{safety} \end{array}$$

Point Values

$$\begin{array}{ccccc} \text{touchdown} & \text{extra point} & \text{field goal} & \text{2-point conversion} & \text{safety} \end{array}$$
$$P = \begin{bmatrix} 6 & 1 & 3 & 2 & 2 \end{bmatrix}$$

You can use matrix multiplication to find the total points scored.

Multiply Matrices You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. When you multiply two matrices $A_{m \times n}$ and $B_{n \times r}$, the resulting matrix AB is an $m \times r$ matrix.

EXAMPLE 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{2 \times 5}$ and $B_{5 \times 4}$

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ 2 \times 5 & & 5 \times 4 & & 2 \times 4 \end{array}$$

The inner dimensions are equal, so the product is defined. Its dimensions are 2×4.

b. $A_{1 \times 3}$ and $B_{4 \times 3}$

$$\begin{array}{ccc} A & \cdot & B \\ 1 \times 3 & & 4 \times 3 \end{array}$$

The inner dimensions are not equal, so the matrix product is not defined.

CHECK Your Progress

1A. $A_{4 \times 6}$ and $B_{6 \times 2}$

1B. $A_{3 \times 2}$ and $B_{3 \times 2}$

The product of two matrices is found by multiplying corresponding columns and rows.

KEY CONCEPT *Multiplying Matrices*

Words The element a_{ij} of AB is the sum of the products of the corresponding elements in row i of A and column j of B.

Symbols $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$

EXAMPLE Multiply Square Matrices

2 Find RS if $R = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$.

$$RS = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 4 & 7 \end{bmatrix}$$

Step 1 Multiply the numbers in the first row of R by the numbers in the first column of S, add the products, and put the result in the first row, first column of RS.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & \\ & \end{bmatrix}$$

Step 2 Follow the same procedure as in Step 1 using the first row and second column numbers. Write the result in the first row, second column.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ & \end{bmatrix}$$

Step 3 Follow the same procedure with the second row and first column numbers. Write the result in the second row, first column.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & \end{bmatrix}$$

Step 4 The procedure is the same for the numbers in the second row, second column.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix}$$

Step 5 Simplify the product matrix.

$$\begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$$

CHECK Your Progress

2. Find UV if $U = \begin{bmatrix} 5 & 9 \\ -3 & -2 \end{bmatrix}$ and $V = \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$.

Study Tip

Multiplying Matrices

To avoid any miscalculations, find the product of the matrices in order as shown in Example 2. It may also help to cover rows or columns not being multiplied as you find elements of the product matrix.

Concepts in Motion

Animation
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Real-World EXAMPLE

3 **SWIM MEET** **At a particular swim meet, 7 points were awarded for each first-place finish, 4 points for each second, and 2 points for each third. Which school won the meet?**

School	First Place	Second Place	Third Place
Central	4	7	3
Franklin	8	9	1
Hayes	10	5	3
Lincoln	3	3	6

Real-World Link

Swim meets consist of racing and diving competitions. There are more than 241,000 high schools that participate each year.

Source: NFHS

Explore The final scores can be found by multiplying the swim results for each school by the points awarded for each first-, second-, and third-place finish.

Plan Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

Results **Points**

$$R = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

Solve Multiply the matrices.

$$RP = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 4(7) + 7(4) + 3(2) \\ 8(7) + 9(4) + 1(2) \\ 10(7) + 5(4) + 3(2) \\ 3(7) + 3(4) + 6(2) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 62 \\ 94 \\ 96 \\ 45 \end{bmatrix}$$ Simplify.

The product matrix shows the scores for Central, Franklin, Hayes, and Lincoln in order. Hayes won the swim meet with a total of 96 points.

Check R is a 4×3 matrix and P is a 3×1 matrix; so their product should be a 4×1 matrix. *Why?*

CHECK Your Progress

3. Refer to the data in Exercises 22–24 on page 174. If the cost of televisions was \$250, DVD players was \$225, video game units was \$149, and CD players was \$75, use matrices to find the total sales for week 1.

Multiplicative Properties Recall that the same properties for real numbers also held true for matrix addition. However, some of these properties do *not* always hold true for matrix multiplication.

EXAMPLE Commutative Property

4 **Find each product if** $P = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$.

a. PQ

$$PQ = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 72 - 42 & -24 + 7 & 16 + 35 \\ -18 + 24 & 6 - 4 & -4 - 20 \\ 0 + 18 & 0 - 3 & 0 - 15 \end{bmatrix} \text{ or } \begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix}$$

b. QP

$$QP = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 72 + 6 + 0 & -63 - 12 + 6 \\ 48 + 2 + 0 & -42 - 4 - 15 \end{bmatrix} \text{ or } \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix}$$

CHECK Your Progress

4. Use $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ -4 & 5 \end{bmatrix}$ to determine whether $AB = BA$ is true for the given matrices.

In Example 4, notice that $PQ \neq QP$. This demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

EXAMPLE Distributive Property

5 **Find each product if** $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix}$, **and** $C = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$.

a. $A(B + C)$

$$A(B + C) = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \right) \quad \text{Substitution}$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 \\ 1 & 10 \end{bmatrix} \quad \text{Add corresponding elements.}$$

$$= \begin{bmatrix} 3(-1) + 2(1) & 3(6) + 2(10) \\ -1(-1) + 4(1) & -1(6) + 4(10) \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix} \quad \text{Multiply columns by rows.}$$

b. $AB + AC$

$$AB + AC = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 3(-2) + 2(6) & 3(5) + 2(7) \\ -1(-2) + 4(6) & -1(5) + 4(7) \end{bmatrix} + \begin{bmatrix} 3(1) + 2(-5) & 3(1) + 2(3) \\ -1(1) + 4(-5) & -1(1) + 4(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 29 \\ 26 & 23 \end{bmatrix} + \begin{bmatrix} -7 & 9 \\ -21 & 11 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix}$$ Add corresponding elements.

CHECK Your Progress

5. Use the matrices $R = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 4 & 6 \\ -2 & 5 \end{bmatrix}$, and $T = \begin{bmatrix} -3 & 7 \\ -4 & 8 \end{bmatrix}$ to determine if $(S + T)R = SR + TR$.

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Notice that in Example 5, $A(B + C) = AB + AC$. This and other examples suggest that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

KEY CONCEPT ***Properties of Matrix Multiplication***

For any matrices A, B, and C for which the matrix products are defined, and any scalar c, the following properties are true.

Associative Property of Matrix Multiplication	$(AB)C = A(BC)$
Associative Property of Scalar Multiplication	$c(AB) = (cA)B = A(cB)$
Left Distributive Property	$C(A + B) = CA + CB$
Right Distributive Property	$(A + B)C = AC + BC$

To show that a property is true for all cases, you must show it is true for the general case. To show that a property is *not* always true, you only need to find one counterexample.

CHECK Your Understanding

Example 1 (p. 177)

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. $A_{3 \times 5} \cdot B_{5 \times 2}$ **2.** $X_{2 \times 3} \cdot Y_{2 \times 3}$ **3.** $R_{3 \times 2} S_{2 \times 22}$

Find each product, if possible.

Example 2 (p. 178)

4. $\begin{bmatrix} 2 & 1 \\ 7 & -5 \end{bmatrix} \cdot \begin{bmatrix} -6 & 3 \\ -2 & -4 \end{bmatrix}$ **5.** $\begin{bmatrix} 10 & -2 \\ -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$

Example 3 (p. 179)

6. $\begin{bmatrix} 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ -2 & 0 \end{bmatrix}$ **7.** $\begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$ **9.** $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

SPORTS **For Exercises 10 and 11, use the table below that shows the number of kids registered for baseball and softball.**

The Westfall Youth Baseball and Softball League charges the following registration fees: ages 7–8, \$45; ages 9–10, \$55; and ages 11–14, \$65.

Team Members		
Age	**Baseball**	**Softball**
7–8	350	280
9–10	320	165
11–14	180	120

10. Write a matrix for the registration fees and a matrix for the number of players.

11. Find the total amount of money the league received from baseball and softball registrations.

Examples 4, 5 (pp. 180–181)

Use $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix}$, **and** $C = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ **to determine whether the following equations are true for the given matrices.**

12. $AB = BA$

13. $A(BC) = (AB)C$

Exercises

HOMEWORK HELP	
For Exercises	**See Examples**
14–19	1
20–27	2, 3
28–30	3
31, 32	4
33, 34	5

Determine whether each matrix product is defined. If so, state the dimensions of the product.

14. $A_{4 \times 3} \cdot B_{3 \times 2}$

15. $X_{2 \times 2} \cdot Y_{2 \times 2}$

16. $P_{1 \times 3} \cdot Q_{4 \times 1}$

17. $R_{1 \times 4} \cdot S_{4 \times 5}$

18. $M_{4 \times 3} \cdot N_{4 \times 3}$

19. $A_{3 \times 1} \cdot B_{1 \times 5}$

Find each product, if possible.

20. $\begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

21. $\begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -7 \end{bmatrix}$

22. $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$

23. $\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix}$

24. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

25. $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

26. $\begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{bmatrix}$

27. $\begin{bmatrix} -4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 \end{bmatrix}$

BUSINESS **For Exercises 28–30, use the table and the following information.**

Solada Fox sells fruit from her three farms. Apples are \$22 a case, peaches are \$25 a case, and apricots are \$18 a case.

Number of Cases in Stock of Each Type of Fruit			
Farm	**Apples**	**Peaches**	**Apricots**
1	290	165	210
2	175	240	190
3	110	75	0

28. Write an inventory matrix for the number of cases for each type of fruit for each farm and a cost matrix for the price per case for each type of fruit.

29. Find the total income of the three fruit farms expressed as a matrix.

30. What is the total income from all three fruit farms combined?

Use $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$, **and scalar** $c = 3$ **to determine whether the following equations are true for the given matrices.**

31. $c(AB) = A(cB)$

32. $(AB)C = (CB)A$

33. $AC + BC = (A + B)C$

34. $C(A + B) = AC + BC$

FUND-RAISING **For Exercises 35 and 36, use the following information.**

Lawrence High School sold wrapping paper and boxed cards for their fund-raising event. The school gets \$1.00 for each roll of wrapping paper sold and \$0.50 for each box of cards sold.

Total Amounts for Each Class

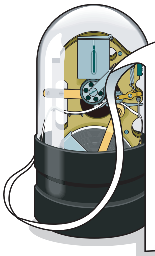
Class	Wrapping Paper	Cards
Freshmen	72	49
Sophomores	68	63
Juniors	90	56
Seniors	86	62

35. Use a matrix to determine which class earned the most money.

36. What is the total amount of money the school made from the fund-raiser?

FINANCE **For Exercises 37–39, use the table below that shows the purchase price and selling price of stock for three companies.**

For a class project, Taini "bought" shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project she "sold" all of her stock.



Company	Purchase Price (per share)	Selling Price (per share)
Utility	\$54.00	\$55.20
Computer	\$48.00	\$58.60
Food	\$60.00	\$61.10

EXTRA PRACTICE
See pages 897, 929.
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37. Organize the data in two matrices and use matrix multiplication to find the total amount she spent for the stock.

38. Write two matrices and use matrix multiplication to find the total amount she received for selling the stock.

39. Use matrix operations to find how much money Taini "made" or "lost" in her project.

H.O.T. Problems

40. OPEN ENDED Give an example of two matrices whose product is a 3×2 matrix.

41. REASONING Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

For any matrix $A_{m \times n}$ for $m \neq n$, A^2 is defined.

42. CHALLENGE Give an example of two matrices A and B for which multiplication is commutative so that $AB = BA$. Explain how you found A and B.

43. CHALLENGE Find the values of a, b, c, and d to make the statement $\begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$ true. If matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ was multiplied by any other two-column matrix, what do you think the result would be?

44. *Writing in Math* Use the data on the Carolina Panthers found on page 177 to explain how matrices can be used in sports statistics. Describe a matrix that represents the total number of points scored in the 2005 season, and an example of another sport where different point values are used in scoring.

STANDARDIZED TEST PRACTICE

45. ACT/SAT What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

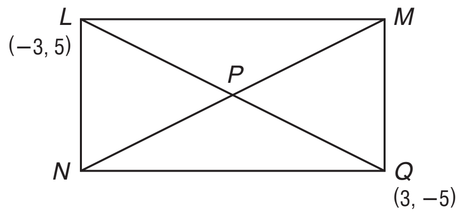
A 1×4

B 3×3

C 4×1

D 4×3

46. REVIEW Rectangle *LMNQ* has diagonals that intersect at point *P*.



Which of the following represents point *P*?

F (1, 1)

G (2, 2)

H (0, 0)

J (−1, −1)

Spiral Review

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. (Lesson 4-2)

47. $3\begin{bmatrix} 4 & -2 \\ -1 & 7 \end{bmatrix}$

48. $[3 \quad 5 \quad 9] + \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$

49. $2\begin{bmatrix} 6 & 3 \\ -8 & -2 \end{bmatrix} - 4\begin{bmatrix} 8 & 1 \\ 3 & -4 \end{bmatrix}$

Solve each equation. (Lesson 4-1)

50. $\begin{bmatrix} 3x + 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 23 \\ -4y - 1 \end{bmatrix}$

51. $\begin{bmatrix} x + 3y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -22 \\ 19 \end{bmatrix}$

52. $\begin{bmatrix} x + 3z \\ -2x + y - z \\ 5y - 7z \end{bmatrix} = \begin{bmatrix} -19 \\ -2 \\ 24 \end{bmatrix}$

53. VACATIONS Mrs. Franklin is planning a family vacation. She bought 8 rolls of film and 2 camera batteries for $23. The next day, her daughter went back and bought 6 more rolls of film and 2 batteries for her camera. This bill was $18. What are the prices of a roll of film and a camera battery? (Lesson 3-2)

Find the x-intercept and the y-intercept of the graph of each equation. Then graph the equation. (Lesson 2-2)

54. $y = 3 - 2x$

55. $x - \frac{1}{2}y = 8$

56. $5x - 2y = 10$

GET READY for the Next Lesson

PREREQUISITE SKILL Graph each set of ordered pairs on a coordinate plane. (Lesson 2-1)

57. {(2, 4), (−1, 3), (0, −2)}

58. {(−3, 5), (−2, −4), (3, −2)}

59. {(−1, 2), (2, 4), (3, −3), (4, −1)}

60. {(−3, 3), (1, 3), (4, 2), (−1, −5)}